Lecture 14

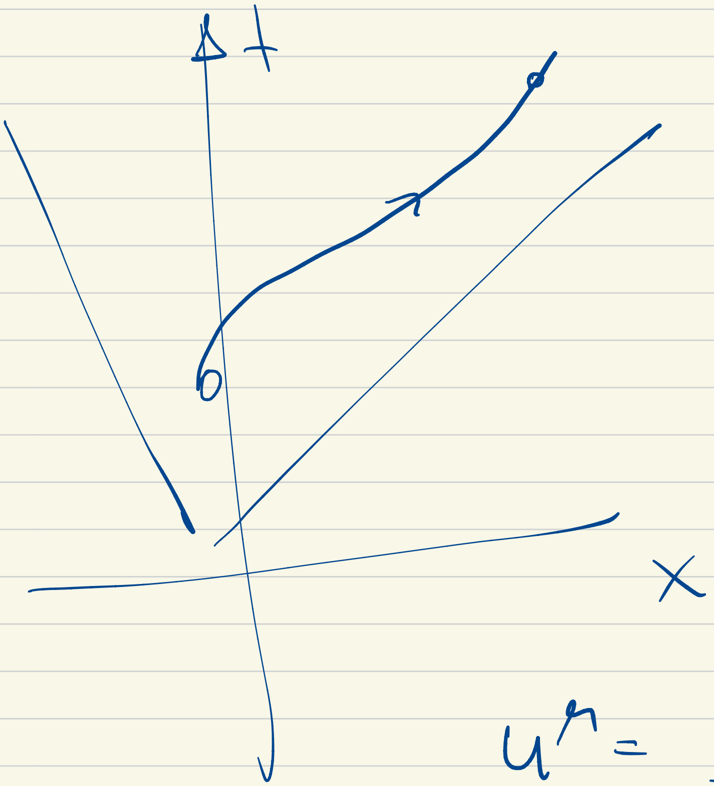
Application of covariant Mechanics and Action principle for CED

1. Recap + Poincare group
2. Motion in constant Magnetic field
3. Radiation of a moving charge
4. Action principle for electromagnetic field.

Poincare group: + Lorentz (6 transformations)
+ Translations
+ T, P - ?

$$\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$$

$$\partial_\mu F^{\mu\nu} = -\frac{1}{c\varepsilon_0} J^\nu$$



$$d\tau = \sqrt{-\dot{x}^\mu \dot{x}_\mu}\, dt$$

proper time

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix}$$

$$u^\mu u_\mu = -c^2 \quad , \quad a^\mu = \frac{du^\mu}{d\tau}$$

$$m^2 a^\mu = \frac{q}{c} F^{\mu\nu} u_\nu$$

[Lorentz force]

$$p^\mu = m u^\mu = \begin{pmatrix} \mathcal{E}/c \\ \vec{p} \end{pmatrix}$$

$$p^\mu p_\mu = -m^2 c^2 \rightarrow \mathcal{E} = mc^2$$

Motion in constant magnetic field

$$\frac{d\vec{p}}{dt} = q(\vec{v} \times \vec{B})$$

$$\frac{d}{dt}\mathcal{E} = 0 \Rightarrow \text{const} \Rightarrow |\vec{v}| = \text{const}$$

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B, \qquad \omega_B = \frac{q\vec{B}}{\gamma m} = \frac{q\vec{B}c^2}{\mathcal{E}}$$

this is called "cyclotron frequency"

$B \parallel B_z$

$$\frac{dv_x}{dt} = v_y \omega_B \qquad \frac{dv_y}{dt} = -v_x \omega_B$$

$$x(t) = -R\cos\omega_B t \qquad y(t) = R\sin\omega_B t$$

$$z(t) = v_z \cdot t$$

Radiation of a moving charge

(Relativistic generalization of Larmor formula.)

Larmor formula (lecture 5).

$$\frac{q^2}{4\pi\varepsilon_0}\dot{v}^2 = \frac{d\mathcal{E}}{dt} \quad \text{(before)}$$

$$dP^\mu = \frac{q^2}{6\pi\varepsilon_0 c^5}\left(\frac{du^\nu}{d\tau}\frac{du_\nu}{d\tau}\right)dx^\mu \rightarrow$$

→ the only expression quadratic in acceleration $\left[a^\mu = \frac{du^\mu}{d\tau}\right]$

$$c^2\frac{dP^0}{dx^0} = \frac{q^2}{6\pi\varepsilon_0 c^3}\frac{du^\nu}{d\tau}\frac{du_\nu}{d\tau} \quad \Leftrightarrow$$

Γ

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \frac{\partial \gamma}{\partial t} = \gamma^3 \dot{\vec{\beta}} \cdot \vec{\beta}$$

$$\frac{\partial}{\partial t} \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix} = c \cdot \begin{pmatrix} (\dot{\beta} \cdot \vec{\beta}) \gamma^2 \\ (\dot{\beta} \cdot \vec{\beta} \cdot \gamma^2) \vec{\beta} + \dot{\vec{\beta}} \end{pmatrix}$$

$$-\gamma^4 (\dot{\beta} \cdot \vec{\beta})^2 + \gamma^4 (\dot{\beta} \cdot \vec{\beta})^2 \cdot \beta^2 + \dot{\vec{\beta}}^2 +$$
$$+ 2 (\dot{\beta} \cdot \vec{\beta})^2 \gamma^2$$

$$= \frac{q^2}{6\pi \varepsilon_0 c^3} \gamma^6 \left[|\dot{\beta}|^2 - |\beta \times \dot{\vec{\beta}}|^2 \right] \qquad (*)$$

Before that, after a very tedious work with Liénard-Wiechert Potentials we derived

$$\frac{d\mathcal{E}}{d\Omega\, dt} = \frac{q^2}{16\pi^2 \varepsilon_0 c} \frac{\left|\vec{n} \times ((\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}})\right|^2}{(1 - \vec{n}\cdot\vec{\beta})^6}$$

if we integrate it over the angles $d\Omega$ we will recover $(*)$.

Energy - momentum (stress) tensor of EM field

$$T^{\mu\nu} = \frac{1}{\mu_0 c^2}\left(F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)$$

$\boxed{\text{Let's check conservation}}$: $F^{\mu\alpha}\partial_\mu F^{\nu}{}_{\alpha} - \frac{1}{2} F^{\alpha\beta}\partial_\nu F_{\alpha\beta} =$

$$= F^{\mu\alpha}\left(\partial_\mu \partial_\nu A^{\alpha} - \cancel{\partial_\mu \partial_\alpha A^{\nu}}\right) -$$

$$-\frac{1}{2} F^{\alpha\beta}\left(\partial_\nu \partial_\alpha A_\beta - \partial_\nu \partial_\beta A_\alpha\right) = 0$$

Lorentz transformations of E.M. fields.

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & cB_z & -cB_y \\ \text{Anti-sym.} & & 0 & cB_x \\ & & & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ & 0 & cB_z & -cB_y \\ & & 0 & cB_x \\ & & & 0 \end{pmatrix}$$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix}$$

$$F^{01'} = \Lambda^0{}_\nu \Lambda^1{}_\mu F^{\mu\nu} =$$

$$= \Lambda^0{}_1 \Lambda^1{}_0 F^{10} + \Lambda^0{}_0 \Lambda^1{}_1 F^{01} =$$

$$= \gamma^2 \beta^2 (-F^{01}) + \gamma^2 F^{01} = F^{01}$$

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel \tag{1}$$
$$\mathbf{E}'_\perp = \gamma\left(\mathbf{E}_\perp + c\beta \times \mathbf{B}\right) \tag{2}$$
$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel \tag{3}$$
$$\mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \frac{1}{c}\beta \times \mathbf{E}\right) \tag{4}$$

$$\begin{aligned} E'_\parallel = E'_x = F'^{01} &\\ &= \Lambda^0_\alpha \Lambda^1_\beta F^{\alpha\beta} \\ &= \Lambda^0_0 \Lambda^1_1 F^{01} + \Lambda^0_1 \Lambda^1_0 F^{10} \\ &= \gamma^2(1-\beta^2)F^{01} \\ &= F^{01} \\ &= E_x = E_\parallel \end{aligned} \tag{5}$$

For the orthogonal component ($j \neq 1$):

$$\begin{aligned} (\mathbf{E}'_\perp)_j = E'_j = F'^{0j} &\\ &= \Lambda^0_\alpha \Lambda^j_\beta F^{\alpha\beta} \\ &= \Lambda^0_0 F^{0j} + \Lambda^0_i F^{ij} \\ &= \gamma(\mathbf{E}_\perp)_j - c\beta\gamma\varepsilon^{1jk} B_k \\ &= \gamma(\mathbf{E}_\perp)_j - c\gamma\varepsilon^{ijk} B_k \beta_i \\ &= \gamma(\mathbf{E}_\perp + c\beta \wedge \mathbf{B})_j \end{aligned} \tag{6}$$

Where we used that $\Lambda^j_\beta = \delta^j_\beta$, and that $\Lambda^0_i = -\delta^1_i \beta\gamma$, $\beta = (\vec{\beta})_x$.
An analogous computation gives the result for the magnetic field $\mathbf{B}$:

$$\begin{aligned} cB'_\parallel = -cB_x = F'^{23} &\\ &= \Lambda^2_\alpha \Lambda^3_\beta F^{\alpha\beta} \\ &= F^{23} \\ &= cB_x = cB_\parallel \end{aligned} \tag{7}$$

$$\begin{aligned} 2c(\mathbf{B}'_\perp)_j = 2cB'_j = \varepsilon_{jlm} F'^{lm} &\\ &= 2\varepsilon_{j1m} F'^{1m} \\ &= 2\varepsilon_{j1m} \Lambda^1_\alpha \Lambda^m_\beta F^{\alpha\beta} \\ &= 2\varepsilon_{j1m}(\Lambda^1_1 F^{1m} + \Lambda^1_0 F^{0m}) \\ &= 2\varepsilon_{j1m}(-\gamma\beta E_m + \gamma c\varepsilon^{1ml} B_l) \\ &= -2\varepsilon_{j1m}\gamma\beta E_m + 2\gamma c B_j \\ &= 2c\gamma(\mathbf{B}'_\perp - \frac{1}{c}\beta \wedge \mathbf{E})_j \end{aligned}$$

Action principle for electromagnetic field and for a charged particle.

$$S = S_f + S_m + S_{mf}$$

Principles:
- Lorentz invariance
- Gauge invariance
- Leading order in fields and derivatives ≈ multipole exp.

$$S_m = S_{particle} = -\int d\lambda \, mc \sqrt{-\dot{x}^\mu \dot{x}_\mu}$$

$\lambda \to f(\lambda)$ gauge inv. = time reparametrisation,
in addition to $A_\mu \to A_\mu + \partial_\mu \alpha$

$$S_{mf} = -\frac{q}{c} \int d\lambda \, A_\mu(x(\lambda)) \, \dot{x}^\mu = -\frac{1}{c} \int A^\mu J_\mu$$

It is gauge invariant : $A^\mu \to A^\mu + \partial^\mu \alpha$,

$$\partial_\mu J^\mu = 0$$

$$\boxed{\begin{gathered} S_g = -\frac{\varepsilon_0}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu} \\ F^{\mu\nu}F_{\mu\nu} \approx E^2 - H^2 \end{gathered}}$$

Also gauge and Lorentz invariant

- Derive Maxwell equations
- Derive Lorentz force:

First variation of $x^\mu(\lambda)$

$\delta S_{mg} =$

integrated by parts

$$-\frac{q}{c}\int d\lambda\, \partial_\nu A_\mu\, \delta x^\nu \cancel{\dot{x}^\mu} + \frac{q}{c}\int d\lambda\, \dot{x}^\nu \cdot$$

$$\cdot\, \partial_\nu A_\mu\, \delta x^\mu$$

check!

δS_m:

$$-\int d\lambda \, \frac{1}{2} \, \frac{\cdot 2 \dot{x}_\mu \cdot \delta \dot{x}^\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}} \cdot m \; c$$

This term also need to integrate by parts:

$$\delta S_m = mc \int d\lambda \, \frac{d}{d\lambda}\left(\frac{\dot{x}_\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}}\right) \delta x^\mu$$

Combining both terms we get the condition that the coefficient of δx^μ vanishes:

$$cm \frac{d}{d\lambda}\left[\frac{\dot{x}_\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}\right] = \left(\partial_\mu A_\nu \dot{x}^\nu - \partial_\nu A_\mu \dot{x}^\nu\right) \frac{q}{c}$$

the right hand side is simply

$$\frac{q}{c} F_{\mu\nu} \dot{x}^\nu$$

Next, we need to fix the gauge for

Time reparametrizations:

choose $\lambda = t$ and multiply both parts by $\frac{1}{\gamma} = \frac{c}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}}$

$$d\tau = \sqrt{1-\beta^2}\, dt \Rightarrow \frac{1}{\gamma}\frac{d}{dt} = \frac{1}{d\tau}$$

$$m \frac{d}{d\tau}\frac{dx_\mu}{d\tau} = \frac{q}{c} F_{\mu\nu} \frac{dx^\nu}{d\tau},$$ in which

we rechognize the relativistic Lorentz force:

$$m a^\mu = \frac{q}{c} F^{\mu\nu} u_\nu$$

The next step is to derive Maxwell equations from variation of the action with respect to A_μ.

- When we vary the action with respect to A_μ we get:

$$\int d^4x \left(\varepsilon_0 \partial_\mu F^{\mu\nu} - \frac{1}{c} J^\nu\right)\delta A_\nu$$

Variation of $F_{\mu\nu} F^{\mu\nu}$ contains four terms, but they all give the same answer (after integrating by parts)

This is the inhomogeneous Maxwell equation!

- Since we work with A_μ.

$\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}$ is automatic.

- To conclude: We derived Maxwell equations, and force from the minimum action principle

- Benefits of the action: symmetries, uniqueness.

Review

Maxwell equations:

$$\text{I} \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\text{II} \quad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\text{III} \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{IV} \quad \vec{\nabla}\cdot\vec{B} = 0$$

III and IV guarantee that we can express $\vec{E}$ and $\vec{B}$ through the potentials:

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

For given $\vec{E}$ and $\vec{B}$ there is ambiguity in $\vec{A}$ and φ called gauge invariance:

$$\begin{cases} \vec{A} \to \vec{A} + \vec{\nabla}\alpha \\ \varphi \to \varphi - \partial_t \alpha \end{cases}$$

leaves $\vec{E}$ and $\vec{B}$ invariant

For example, adding arbitrary constants to the potentials $[\alpha(\vec{x}, t) = C_i x_i + C_0 t]$ does not have any physical consequences.

We can fix the gauge by using Lorentz gauge condition:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$$

Then I and II Maxwell eqns become:

$$\Box \varphi = \frac{\rho}{\varepsilon_0}$$

$$\Box \vec{A} = \mu_0 \vec{J}$$

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

[III and IV are satisfied automatically]

- It is often easier to find φ and $\vec{A}$ in a given problem and then derive $\vec{E}$ and $\vec{B}$ using

$$\begin{cases} \vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

- We considered the following types of problems:

	Static Δ	Dynamical $\Box$
homogeneous $\rho, \vec{j} = 0$	trivial	Plane waves
inhomogeneous $\rho, \vec{j} \neq 0$	• Boundary problems • Multipole exp.	• Radiation of EM waves • Multipole exp.

Boundary problems

Electric

$$\Delta \varphi = -\frac{\rho}{\varepsilon_0}$$

Magnetic

$$\Delta \vec{A} = -\vec{J}\mu_0$$

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{J} = 0$$

Boundary problems

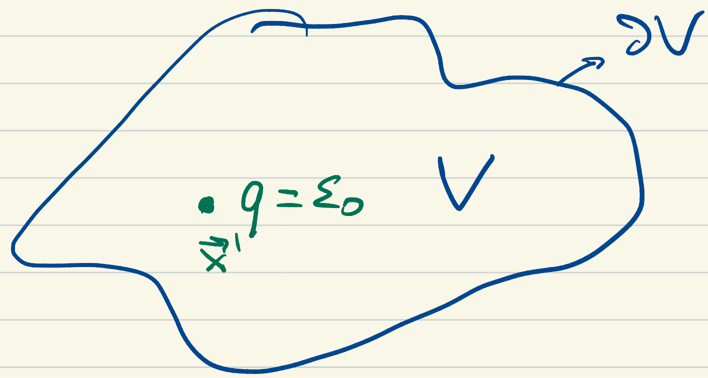
Image Method
("Simple" boundary and sources)

Greens function
("Generic" boundary and sources)

Greens function:

$$\Delta G(\vec{x}, \vec{x}') = -\delta^3(\vec{x} - \vec{x}') \quad (*)$$

+ Boundary conditions



(*) is satisfied only inside V

$$G = \frac{1}{4\pi |\vec{x} - \vec{x}'|} + F, \quad \Delta F = 0$$

inside V

Physical meaning of G:

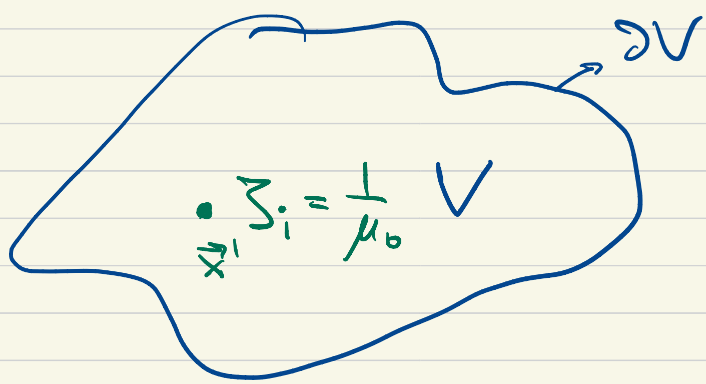
Potential of a charge $q = \varepsilon_0$ at a point $\vec{x}'$, plus of the charges outside of V or on the boundary,

that together produce the
needed boundary conditions.

Works equivalently for magnetic case:

Vector Potential of a ~~charge~~ current $\cancel{q = \varepsilon_0}$ $J_i = \frac{1}{\mu_0}$ at a
point $\vec{x}'$, plus of the ~~charges~~ currents
outside of V or on the boundary,
that together produce the
needed boundary conditions.

$$\Delta A_i = - J_i \mu_0$$



Commonly used B.C. are Neumann or Dirichlet:

$$G^D\big|_{\partial V} = 0 \qquad \varphi\big|_{\partial V} = \varphi_0 :$$

$$\varphi(\vec{x}) = \frac{1}{\varepsilon_0}\int_V d^3x' \, G^D(\vec{x},\vec{x}')\rho(\vec{x}') - \oint_{\partial V} d^2\vec{S}\cdot\vec{\nabla}\, G^D(\vec{x},\vec{x}')\,\varphi_0(\vec{x}')$$

$\vec{x} \in V$

$\vec{x}' \in \partial V$

Neumann:

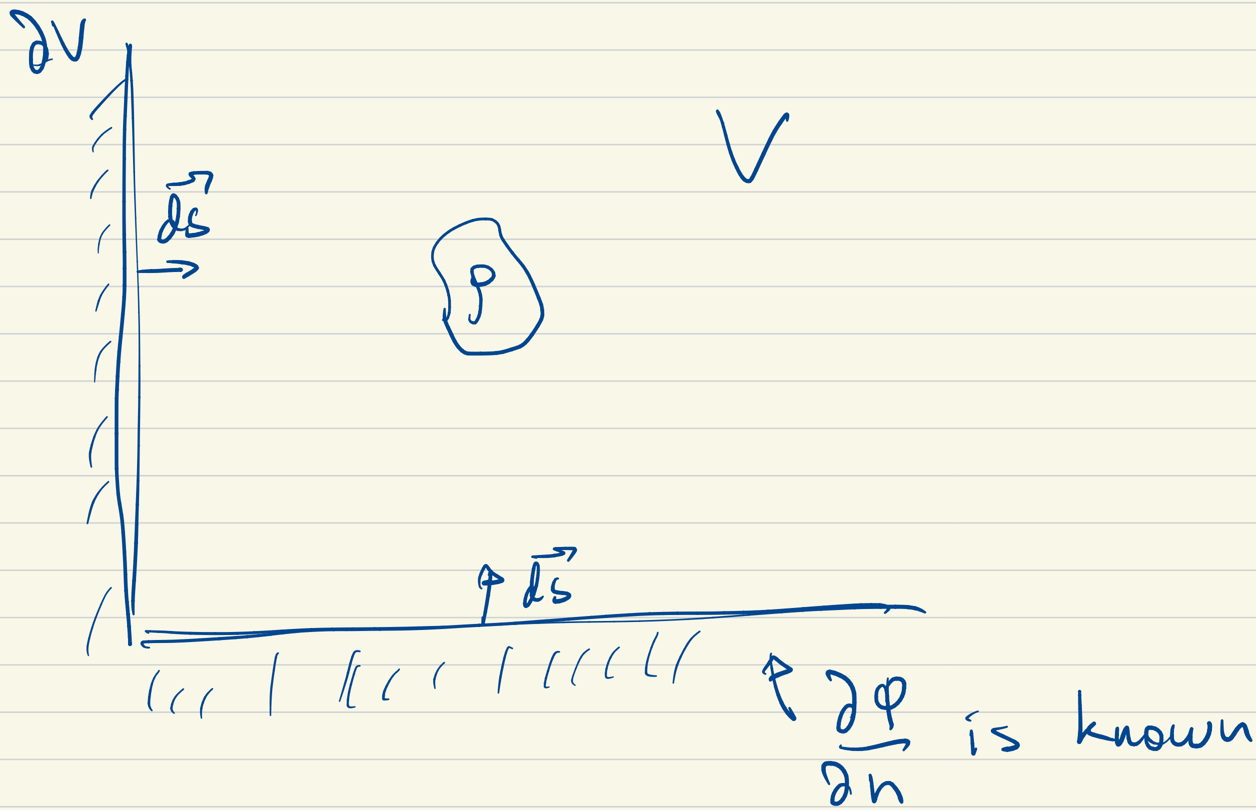
$$\frac{\partial}{\partial n'} G^N(\vec{x},\vec{x}')\Big|_{\partial V} = 0 \qquad \frac{\partial}{\partial n}\varphi\Big|_{\partial V} = g :$$

$$\varphi(\vec{x}) = \frac{1}{\varepsilon_0}\int_V d^3x' \, G^N(\vec{x},\vec{x}')\rho(\vec{x}') + \oint_{\partial V} d^2x' \, g(x')\, G^N(\vec{x},\vec{x}')$$

$x \in V$

$\vec{x}' \in \partial V$

"Here we assumed an open boundary"

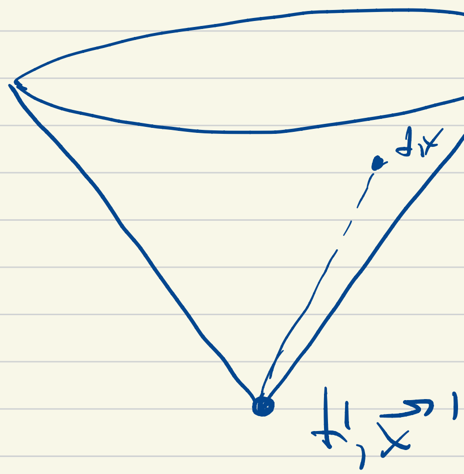


- Radiation of EM waves:

$$\Box G_{ret}(\vec{x}, t, \vec{x}', t') = \delta(t-t')\,\delta^3(\vec{x}-\vec{x}')$$

$$G_{ret} = \frac{\delta\left(t-t'+\frac{|\vec{x}-\vec{x}'|}{c}\right)}{4\pi\,|\vec{x}-\vec{x}'|}$$

localized on the lightcone:



$$(t - t') \cdot c = |\vec{x} - \vec{x}'|$$

$$t > t'$$

Physical meaning: Electric (Magnetic) field produced by a point-like instantaneous charge (current)

$$\Box \varphi = \frac{\rho}{\varepsilon_0} \qquad \Box \vec{A} = \mu_0 \vec{J}$$

$$\varphi(\vec{x}, t) = \frac{1}{\varepsilon_0} \int \rho(\vec{x}', t') \cdot G_{ret}(\vec{x}, t, \vec{x}', t')$$

In medium:

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

polarization

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \equiv \varepsilon \vec{E}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \equiv \frac{\vec{B}}{\mu}$$

magnetization

+ Matching conditions

• We found the multipole expansion in the static case:

summed over

$$\varphi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{n!} Q_{i_1 \dots i_n} \frac{x_{i_1} \dots x_{i_n}}{|\vec{x}|^{2n+1}}$$

$i_k = 1 .. 3$

$Q_{i_1 \dots i_n}$ = 2^n-pole moment tensor of the charge density:

$$Q_{i_1 \dots i_n} = \int d^3x' \rho(x') T_{i_1 \dots i_n}(x')$$

$$T_{i_1 \dots i_n} = (2n-1)!! \, x_{i_1} x_{i_2} \dots x_{i_n} - A_{i_1 \dots i_n}$$

$A_{i_1 \dots i_n}$ contains δ_{ij}'s so that to make $T_{i_1 \dots i_n}$ traceless with respect to all indices.